

TABLE I

p_0	b	n	$P(Y_n/n \geq 0)$	Equation (9)	$\exp(c(\theta_0)n)$
0.1	0.1	5	2.86×10^{-2}	1.41×10^{-2}	7.78×10^{-2}
		25	1.26×10^{-7}	4.59×10^{-8}	2.84×10^{-6}
		50	4.33×10^{-14}	4.62×10^{-14}	8.08×10^{-12}
0.3	0.2	25	7.52×10^{-3}	6.63×10^{-3}	1.13×10^{-1}
		50	2.11×10^{-4}	2.65×10^{-4}	1.28×10^{-2}
		100	1.06×10^{-6}	1.21×10^{-6}	1.64×10^{-4}
0.01	0.005	4	3.46×10^{-4}	6.38×10^{-4}	1.57×10^{-3}
		8	2.64×10^{-7}	3.54×10^{-7}	2.46×10^{-6}
		16	2.68×10^{-13}	3.08×10^{-13}	6.05×10^{-12}

For a numerical example, suppose that $X_i = \pm 1$ with $P(X_i = +1) = p_0 - bp_0$ with $p_0 < 1/2$ and $b \leq p_0$, and Ψ is a uniform random variable on $[0,1]$. After a little work it can be shown that

$$P(Y_n/n \geq 0) = \frac{1}{b} \sum_{k=\lfloor n/2 \rfloor}^n \binom{n}{k} [\beta_{p_0}(k+1, n-k+1) - \beta_{p_0-b}(k+1, n-k+1)],$$

where $\beta_x(a, b)$ is the incomplete beta function. This formula can be numerically evaluated using the BETDF subroutine from the IMSL library. We also find that $\theta_0 = \frac{1}{2} \log((1-p_0)/p_0)$, $c(\theta_0) = \log(2\sqrt{p_0(1-p_0)})$, $\sigma^2 = 1$ and $\tau(\theta_0) = b(1-2p_0)/[2p_0(1-p_0)]$. Table I compares some numerical values of the exact value $P(Y_n/n \geq 0)$, the asymptotically sharp approximation given in (9), and the crude exponential approximation $\exp(c(\theta_0)n)$. A similar comparison is carried out in the i.i.d. setting in ([7, pp. 129–131]).

A practical situation where conditional i.i.d. sums arise is in the analysis of the correlator receiver for direct sequence spread spectrum, multiple access communications systems. In this application, the random phases and timing delays of interfering spread spectrum signals play the role of the "nuisance variable" Ψ . A more detailed large deviations analysis of this receiver is given in [11].

IV. DISCUSSION

We note that finding the asymptotics of $M_n(\theta_0)$ can in of itself be a nontrivial problem. Our philosophy has been to assume that knowledge of the moment generating function sequence is complete. In the setting of the first example and in more general cases of the third, this can be a nontrivial task, even though the logarithmic behavior is known.

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The Asymptotic Risk in a Signal Parameter Estimation Problem

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Abstract—In estimating the unknown location of a rectangular signal observed with white noise, the asymptotic risks of three important estimators are compared under L_1/L_2 losses. A different numerical scheme is used to improve the accuracy of Ibragimov/Hasminskii's result, which also leads to further information and numerical comparisons about the problem.

Index Terms—Rectangular signal, Bayes/minimax risks, squared/absolute error losses, MLE.

I. INTRODUCTION

Consider an observed signal of the form

$$dr(t) = s(t - \theta)dt + \sigma dB(t), \quad t \in (0, T + 1), \quad (1)$$

where $B(\cdot)$ is Brownian motion, $\theta \in (0, T)$ is an unknown shift parameter, and s is the rectangular signal with $s(\tau) = X_{[0,1]}(\tau)$. The objective is to estimate θ under normalized squared error loss, $L_2 = \sigma^{-4}(d - \theta)^2$, or under normalized absolute loss, $L_1 = \sigma^{-2}|d - \theta|$.

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Of interest here is precise evaluation of the asymptotic minimax risk as $\sigma \rightarrow 0$. This is asymptotically the same as the Bayes risk for a uniform prior on $[0, T]$. Some comparisons will be made between this minimax risk and the risk of some natural nonoptimal estimators.

This problem was analyzed in Ibragimov and Hasminskii [3, ch. 7]. Let $M_{i,\sigma}$ denote the minimax risk under L_i . (Throughout T will be assumed fixed and known.) It is claimed in [3, p. 345] that

$$\lim_{\sigma \rightarrow 0} M_{2,\sigma} \approx (4.875 \pm 0.125). \quad (2)$$

This claim was based on asymptotic theory followed by numerical computations including a Monte Carlo simulation. We will provide a more precise evaluation for this limit. Our analysis begins somewhat differently from that in [3] and culminates in a somewhat different Monte Carlo simulation. We find

$$M_2 = \lim_{\sigma \rightarrow 0} M_{2,\sigma} \approx 4.762 \pm 0.015, \quad (3)$$

which, of course, agrees with (2). (The error term here is the Monte Carlo estimate for the standard deviation of the simulation risk. It ignores the possible bias of this simulated risk. This bias will be discussed later.) [3] did not consider the loss L_1 in this context. We find

$$M_1 = \lim_{\sigma \rightarrow 0} M_{1,\sigma} \approx 1.383 \pm 0.0018. \quad (4)$$

(The analysis in [3, ch. 7] also shows how the same idea can be directly applied when s is any signal form having a finite number of discontinuities.)

The theoretical basis for our evaluations is explained in Section II. Section III reports numerical results which supplement (3) and (4). It contains a comparison of the asymptotic risks under both L_1 and L_2 of three different estimators: the maximum likelihood estimate, the posterior median (which is minimax for L_1) and the posterior mean (which is minimax for L_2).

II. ASYMPTOTIC THEORY

When studying asymptotic properties, it is sufficient to fix T at any convenient value. Hence, let $T = 1$ in what follows.

Consider a reparametrization in which

$$\theta = 2(\phi - 1/2)/\sigma^2, \quad \theta \in (-1/\sigma^2, 1/\sigma^2).$$

For fixed σ , let

$$Z_t = \sigma^{-2} \int [s(\nu - 1/2 - \sigma^2 t/2) - s(\nu - 1/2)] dr(\nu), \\ t \in (-1/\sigma^2, 1/\sigma^2).$$

Note that $\{Z_t\}$ is the Gaussian process with

$$E_\phi(Z_t) = (|\phi| - |\phi - t|)/2 \equiv \mu_\phi(t), \\ \text{var}_\phi(Z_t) = |t|, \\ \text{cov}_\phi(Z_t, Z_w) = \begin{cases} \min(|t|, |w|), & \text{if } \text{sign}(t) = \text{sign}(w) \text{ and } |t|, |w| > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Consequently, Z_t can be represented as

$$Z_t = \begin{cases} V_{-t} + \mu_\phi(t), & 1/\phi^2 < t < 0, \\ W_t + \mu_\phi(t), & 1/\sigma^2 > t \geq 0, \end{cases}$$

where $\{W_t\}$ and $\{V_t\}$ are independent copies of standard Brownian motion on $[0, 1/\sigma^2]$. Note that the range of t depends on σ^2 and increases to $(-\infty, \infty)$, but the distribution of $\{Z_t\}$ does not otherwise depend on t .

Let $\phi_1, \phi_2 \in (-1/\sigma^2, 1/\sigma^2)$. Think of the statistical problem of choosing whether $\phi = \phi_1$ or ϕ_2 after observing $dr(\cdot)$ given by (1) with $\theta = \sigma^2 \phi_1/2 + 1/2$ or $\sigma^2 \phi_2/2 + 1/2$. It is easy to check, and well known, that $Z_{\phi_1} - Z_{\phi_2}$ is a (minimal) sufficient statistic for the problem. Consequently, $\{Z_t\}$ is pairwise sufficient for any pair ϕ_1, ϕ_2 . It follows that $\{Z_t\}$ is also sufficient for the original statistical problem having $\phi \in (-1/\phi^2, 1/\phi^2)$; see [1].

Estimation of θ by d under loss L_i is equivalent to estimation of ϕ by $\delta = 2(d - 1/2)/\sigma^2$ under the loss $L'_i = |\delta - \phi|^i/2^i$. In summary, *the problem of estimating θ under loss L_i after observation of $dr(\cdot)$ is equivalent to that of estimating ϕ under loss L'_i after observation of $Z_t, t \in (-1/\sigma^2, 1/\sigma^2)$.*

It is now clear that the asymptotic form of the original signal parameter estimation problem is equivalent to the problem of observing $\{Z_t\}$ for $t \in (-\infty, \infty)$ with unknown parameter $\phi \in (-\infty, \infty)$. A local version of this asymptotic equivalence was already established by a different method in [3]. The explicit construction enables one to draw certain conclusions about the nonasymptotic problem that do not logically follow from the local asymptotic equivalence theory in [3]. For example, *the minimax risk under the losses L_1 or L_2 for the problem of estimating θ increases as σ decreases to 0, and its limit is the global minimax value for the problem of estimating $\phi \in (-\infty, \infty)$ based on observation of Z_t .*

Let $f_\phi(\{z_t\})$ denote the density of $\{Z_t\}$ under $\phi \in (-\infty, \infty)$ with respect to the distribution of $\{Z_t\}$ under $\phi = 0$. It can be checked that $f_\phi(\{z_t\}) = \exp(z_\phi)$. Note that the distribution of $Z_t - Z_\phi$ under parameter ϕ is independent of ϕ . Hence, the problem of estimating ϕ under loss L_1 is location invariant. (A maximal invariant in the sample space is $\{(z_t - z_\phi) : t \in (-\infty, \infty)\}$ where $\hat{\sigma}$ is the maximum likelihood estimation—i.e., $z_\phi = \max z_t$.) It follows that minimax estimator is the Pitman estimator. For $i = 2$ this is given by

$$\delta_i(z_t) = \frac{\int_{-\infty}^{\infty} \phi \exp(z_\phi) d\phi}{\int_{-\infty}^{\infty} \exp(z_\phi) d\phi} \quad (6)$$

(see also [3, formula (2.17), p. 338]), which is the posterior mean under a uniform prior. (See, e.g., [2, p. 405].)

For $i = 1$, the estimator is the posterior median under the prior, i.e.,

$$\int_{-\infty}^{\delta_1} \exp(z_\phi) d\phi = \int_{\delta_1}^{\infty} \exp(z_\phi) d\phi. \quad (7)$$

These estimators will have constant risk since they are invariant. Hence, the minimax value is

$$M_i = 2^{-i} E_0 \left(|\delta_i(Z_t)|^i \right). \quad (8)$$

It is apparently impossible to analytically obtain the value in (8). Numerical methods seem to be needed.

In order to evaluate (8) in the case $i = 2$, Ibragimov and Hasminskii adopted a sophisticated scheme which represents the numerator and the denominator of (6) as limiting solutions to a system of stochastic differential equations. They then used a Monte Carlo simulation to estimate the joint distribution of these limiting values and hence to estimate (8). We take a different approach which is both more elementary and more precise. (See Remark 1 concerning precision.) Furthermore, it is easy to adapt our approach to accommodate other estimators, such as δ_1 . Our method begins by approximating the integrals in (6) by finite Riemann sums. These sums are random, and so their distribution is then simulated. Because the quantity of interest, δ_2 , involves a ratio of terms whose joint

distribution is unknown, we elected to use a relatively straightforward simulation scheme rather than risk introducing further bias.

Consider δ_2 : The integrals in (6) can be approximated by the trapezoid rule. Since $z_0 = 0$ we get

$$\delta_2(z_t) \approx \frac{\sum_{j=1}^{\infty} (j\Delta) \exp(z_{j\Delta}) - \sum_{j=1}^{\infty} (j\Delta) \exp(z_{-j\Delta})}{1 + \sum_{j=1}^{\infty} \exp(z_{j\Delta}) + \sum_{j=1}^{\infty} \exp(z_{-j\Delta})} \equiv \delta_{2,\Delta}(z_t). \quad (9)$$

A further claim can be made. $\delta_{2,\Delta}$ is the formal Bayes rule relative to the uniform prior on the points $\phi = j\Delta, j = 0, \pm 1, \pm 2, \dots$. As such, it has constant risk and is minimax when the parameter space is restricted in this way. The minimax risk for a restricted parameter space is at most that of the unrestricted space. Hence,

$$2^{-2} E_0(\delta_{2,\Delta}^2) \leq 2^{-2} E_0(\delta_2^2(Z_t)) = M_2. \quad (10)$$

It remains to evaluate the left hand code of (10) by numerical methods. To this end the infinite sums in (9) must be approximated by finite sums. Recall that $Z_t = W_t$, for $t \geq 0$, and $Z_t = -V_{-t}$, for $t \leq 0$, as noted after (5). So for large k

$$M_2 \approx 4^{-1} \Delta^2 E \left\{ \left[\frac{N_{1,\Delta}(k) - N_{2,\Delta}(k)}{1 + D_{1,\Delta}(k) + D_{2,\Delta}(k)} \right]^2 \right\}, \quad (11)$$

where

$$D_{1,\Delta}(k) = \sum_{j=1}^k \exp(W_{j\Delta}), \quad N_{1,\Delta}(k) = \sum_{j=1}^k j \cdot \exp(W_{j\Delta}),$$

$$D_{2,\Delta}(k) = \sum_{j=1}^k \exp(V_{-j\Delta}), \quad N_{2,\Delta}(k) = \sum_{j=1}^k j \cdot \exp(V_{-j\Delta}),$$

and the pair $(D_{2,\Delta}(k), N_{2,\Delta}(k))$ is independent of $(D_{1,\Delta}(k), N_{1,\Delta}(k))$ and has the same distribution. Finally, note that

$$W_{(j+1)\Delta} = W_{j\Delta} + U_j, \quad j = 0, 1, \dots, k, \quad (12.1)$$

where $W_0 = 0$, and $U_j \sim N(-\Delta/2, \Delta)$ independently of $W_{j\Delta}$. Similarly,

$$V_{-(j+1)\Delta} = V_{-j\Delta} + U'_j, \quad j = 0, 1, \dots, k \quad (12.2)$$

where $V_0 = 0$, and $U'_j \sim N(-\Delta/2, \Delta)$ is independent of $V_{-j\Delta}$. This enables a convenient iterative Monte Carlo calculation of D and N via the scheme

$$D_{i,\Delta}(0) = 0, \quad N_{i,\Delta}(0) = 0, \quad L_{i,\Delta}(0) = 1;$$

$$L_{i,\Delta}(j+1) = \exp(U_{j+1}) \cdot L_{i,\Delta}(j);$$

$$D_{i,\Delta}(j+1) = D_{i,\Delta}(j) + L_{i,\Delta}(j+1); \quad (13)$$

$$N_{i,\Delta}(j+1) = N_{i,\Delta}(j) + (j+1) \cdot L_{i,\Delta}(j+1), \quad j = 0, 1, \dots, k.$$

Numerical results are described in the following section.

Now Consider δ_1 : From (8) and (13), it can be seen that δ_1 is well approximated by $\delta_{1,\Delta}(k) = \Delta m$ when m depends on Δ, k , and $\{U_i\}$ and is determined by the following procedure:

IF $|D_{2,\Delta}(k) - D_{1,\Delta}(k)| \leq 1$,

THEN $m = 0$.

IF $D_{1,\Delta}(k) - D_{2,\Delta}(k) > 1$,

THEN $D_{1,\Delta}(m) \geq \frac{D_{1,\Delta}(k) - D_{2,\Delta}(k) - 1}{2} > D_{1,\Delta}(m-1)$.

IF $D_{2,\Delta}(k) - D_{1,\Delta}(k) > 1$,

THEN $m < 0$ AND

$$D_{2,\Delta}(-m) \geq \frac{D_{2,\Delta}(k) - D_{1,\Delta}(k) - 1}{2} > D_{2,\Delta}(1 - (m-1)). \quad (14)$$

As in (10), it is the case here that $2^{-1} E_0(\delta_{1,\Delta}) \leq M_1$, and the two are approximately equal when Δ is small.

Remark 1: The algorithm described in (11) and (13) for approximating M_2 is different from that used in [3], but a comparison is possible. Although the motivation and the derivation are different from ours, it appears that the numerical scheme adopted there is virtually equivalent to that which would result from using (11) and (13) with $\exp(U_j)$ replaced by $1 + U_j$. Thus, our scheme should be slightly more precise for a given Δ , but either scheme should converge to M_2 as $\Delta \rightarrow 0$ and $k \rightarrow \infty$.

Remark 2: There are two sources of bias in the preceding simulation schemes for estimating M_1 and M_2 . Consider, for example, M_2 . The first source is the approximation (9). The Riemann sums in the numerator and denominator are each negatively biased estimators of the respective integrals in (6), and as noted in (10) their ratio is also a negatively biased estimate of M_2 . It can be shown via the reasoning leading to (10) that the magnitude of this bias is less than $3\Delta^2$ and hence can easily be made quite small. The second source of bias is the truncation, implicit in (11), of the infinite sums in (9) to be finite sums over $1 \leq j \leq k$. The magnitude of this bias is much harder to estimate since $E(\exp(z_t)) = 1$ for every t so that $E(\sum_{j=k+1}^{\infty} \exp(z_t)) = \infty$, etc. Our only substantial evidence that this bias is small is derived from the numerical results in Section III. In Table II, doubling k affects the simulation estimate by at most 1% of its value and usually by much less.

Another estimator of interest is the *maximum likelihood estimator* (MLE). In the original problem (1) this can easily be seen to be the value $\hat{\theta}$ which maximizes $\int s(t-\theta) dr(t)$. In the equivalent formulation, (5), the MLE is the value $\hat{\phi}$ that maximizes Z_ϕ . The L_2' risk of $\hat{\phi}$ is found in [3] to be (see [4])

$$E(L_2'(\phi, \hat{\phi})) = 6.5 \text{ (exactly)}. \quad (15)$$

Their results also yield

$$E(L_1'(\phi, \hat{\phi})) = 1.5 \text{ (exactly)}. \quad (16)$$

Equation VII.3.11 of [3] establishes that

$$\zeta(\lambda) = \int_0^{\infty} \exp(-\lambda t) P(|\hat{\phi}| > t) dt$$

$$= 16 \left(1 + (1 + 8\lambda)^{1/2} \right)^{-2}$$

$$- 8 \left[\left(1 + (1 + 8\lambda)^{1/2} \right) \left(3 + (1 + 8\lambda)^{1/2} \right) \right]^{-1},$$

$$\text{so } E(L_1'(\phi, \hat{\phi})) = \zeta(0)/2 = 1.5.$$

Remark 3: [3] shows that when properly normalized the problem described in (5) is the asymptotic local limit for any signal parameter estimation problem of the form (1) in which $s(\cdot)$ is a known signal having compact support and possessing a finite number of discontinuities. Hence, the numerical results reported in Table I also apply in such cases. It should be emphasized that the results obtained in this generality concern only local asymptotic properties. They do not directly yield statements about the limiting global minimax risk for rectangular s , nor do they yield the stronger statement that the normalized minimax risk for given σ and $T = 1$ is bounded above by its asymptotic value.

TABLE I
RISKS OF $\delta_1, \delta_2, \hat{\theta}$ UNDER L_1 AND L_2

	Risk in L_1 loss	Risk in L_2 loss	Δ	k
δ_1	$M_1 = 1.3827 \pm 0.0018$	5.1601 ± 0.0183	0.01	5000
δ_2	1.4446 ± 0.0016	$M_2 = 4.7622 \pm 0.0147$	0.01	5000
$\hat{\theta}$	1.5	6.5		

TABLE II
RISKS UNDER L_1 AND L_2

Δ	Estimator	$k\Delta$	L_1 Est. Risk \pm S.D.	L_2 Est. Risk \pm S.D.
0.5	δ_2	50	1.4193 ± 0.0016	4.7056 ± 0.0148
		25	1.4202 ± 0.0016	4.7242 ± 0.0150
	δ_1	50	1.3124 ± 0.0019	5.2444 ± 0.0186
		25	1.3133 ± 0.0019	5.2700 ± 0.0190
0.25	δ_2	50	1.4403 ± 0.0016	4.7746 ± 0.0150
		25	1.4426 ± 0.0016	4.7906 ± 0.0152
	δ_1	50	1.3651 ± 0.0018	5.2271 ± 0.0188
		25	1.3670 ± 0.0018	5.2331 ± 0.0188
0.1	δ_2	50	1.4456 ± 0.0016	4.7918 ± 0.0151
		25	1.4474 ± 0.0016	4.7978 ± 0.0151
	δ_1	50	1.3815 ± 0.0018	5.2065 ± 0.0189
		25	1.3827 ± 0.0018	5.1994 ± 0.0187
0.01	δ_2	50	1.4446 ± 0.0016	4.7622 ± 0.0147
		25	1.4475 ± 0.0016	4.7965 ± 0.0148
	δ_1	50	1.3827 ± 0.0018	5.1601 ± 0.0183
		25	1.3859 ± 0.0018	5.2023 ± 0.0185

III. NUMERICAL RESULTS

The L_1 and L_2 risks of δ_1 and δ_2 were computed according to the simulation scheme described in Section II. These risks are given in Table I, along with the values of k and Δ , which were used. Each entry involved 10^6 iterations of the entire scheme. The standard errors shown are the square roots of the conventional unbiased estimators of variance obtained in the simulation trials. (We do not have any satisfactory theoretical results concerning the precision of the simulated risks or of their estimated standard deviations). The risk of the MLE, $\hat{\theta}$ is given in (16), (17).

Note that the risks of δ_1 and δ_2 are rather similar, and for squared error loss both are much better than the MLE.

In order to ascertain the effect of altering Δ and k , and to decide on apparently satisfactory values for use in Table I, several other simulations were conducted. A few of these are reported in Table II. The entries labeled S.D. in the 4th and 5th columns of Table II are the estimated standard deviations. As before, each entry of the table involved 10^6 iterations.

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A Note on Rearrangements, Spectral Concentration, and the Zero-Order Prolate Spheroidal Wavefunction

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Abstract—If the measure of the support of a function f is small, its symmetric decreasing rearrangement f^* is more nearly bandlimited to low frequencies than f , while their norms are equal. An immediate corollary is that the time-limited zero-order prolate spheroidal wavefunction is the extremal function for a new optimization problem involving time- and bandlimiting. The result has an application in exploration seismology.

Index Terms—Symmetric decreasing rearrangement, uncertainty principle, smoothing, sparsity constraints, prolate spheroidal wavefunctions.

I. INTRODUCTION

A. The Question

Let f and \hat{f} denote a function and its Fourier transform, respectively, and measure the concentration in the low frequencies $[-W/2, W/2]$ by

$$c_W(f) \equiv \frac{\int_{-W/2}^{W/2} |f|^2 d\omega}{\int_{-\infty}^{\infty} |\hat{f}|^2 d\omega}.$$

Here is a new extremal property of the zero-order prolate spheroidal wavefunction ϕ_0 . Let the time-bandwidth product $WT \leq 4/5$. Among all functions f of limited support T ($\text{meas}(\text{supp}(f)) \leq T$), $c_W(f)$ is maximized by a function ϕ_0 supported on the interval $I_T \equiv [-T/2, T/2]$ and ϕ_0 is the restriction $\phi_0 = \phi_0 \cdot 1_T$ of ϕ_0 to that interval:

$$\sup_{\text{meas}(\text{supp } f) \leq T} c_W(f) = c_W(\tilde{\phi}_0). \tag{1.1}$$

In short, $\tilde{\phi}_0$ is the most nearly bandlimited function of all functions supported on set of measure $\leq T$.

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